Minutes of General Meeting of the Victorian Institute of Engineers, held in the Radio Theatre, Melbourne Technical College, Melbourne.

In the absence of the President (Mr. W. H. Dobson), the Council nominated, and it was agreed that, Mr. L. L. Pemberton occupy the chair. (Both Vice-Presidents were, unfortunately, unable to attend.)

Apologies for non-attendance were received from Messrs. W. H. Dobson, E. J. L. Bremner, V. R. McKay, Geo. D. Thompson, G. E. Gamble and E. J. McDonald.

The minutes of meeting held 26th June, 1947, were read and approved. Correspondence was reviewed, and general announcements made.

A ballot for election as Member of Mr. N. G. Daws, and Junior Member of Mr. R. N. Cox, was conducted.

A Lecture was given (freely illustrated with diagrams and specimens) by Mr. N. Sag, Degree Mech.E. (Budapest University), Senior Lecturer in Machine Design, Melbourne Technical College, on "Principles of Gear Tooth Correction."

Discussion ensued.

Ballot was declared in the affirmative for both candidates.

Meeting concluded with light supper.

PAPER

Fundamental Principles of Modern Gear Design Practice

By

Nicholas Sag,
Degree Mech.E. (Budapest Univ.),
Senior Lecturer in Machine Design (Melb. Tech. College).

1947.

Introduction

To ensure constant gear ratio at any instant, tooth profiles used for gearing must fulfil certain conditions.

In Fig. 1, O₁ and O₂ represent the centres of two mating gears, and the tooth profiles are shown in contact at point Q. The momentary motion at this instant is in the direction of the common normal C₁C₂ and the instantaneous velocity in this direction is denoted by V. Then the angular velocities can be expressed as

$$\omega_1 = \frac{V}{r_1} \text{ and } \omega_2 = \frac{V}{r_2}$$

where $r_1$ and $r_2$ are the distances of centres $O_1$ and $O_2$ from the common normal of the tooth profiles. The condition to be satisfied is

$$\frac{\omega_1}{\omega_2} = \text{const.}$$

But

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} = \frac{O_2P}{O_1P} = \text{const.}$$
Hence the intersection (P) of the common normal with the line of centres O1O2 must be a fixed point, i.e.: "For constant gear ratio, the tooth profiles must be such that the common normals at all points of contact pass through a fixed point P on the line of centres."

This point P is called the pitch point, and the locus of all points of contact is the line of action. The line of action must pass through point P.

The tooth profiles of a basic rack from which mating gears are generated must have a tooth profile symmetrical with respect to the pitch point. This will be fulfilled if the line of action is also symmetrical with respect to the pitch point. The cycloidal and involute gears have such profiles.

The cycloidal gears, due to many limitations, such as non-linear basic rack profile, inflexibility in production etc., are not suitable for general use. The involute tooth profile has many advantages, especially as far as production facilities are concerned, which far outweigh the drawback resulting from sliding contact. Moreover, due to the straight line basic rack profile, it can be accurately produced so that the friction loss due to tooth friction can be reduced to a very small proportion of the total power transmitted. Also, unlike the cycloid, the shape of the involute tooth profile depends only on the size of the base circle, so that involute gears will mate correctly at varying centre distances. (This property is fully utilised in modern gear design practice.) All involute gears of the same base
pitch can be generated from the same basic rack, independent of the position of
the pitch point relative to the gear centre.

The aim of the British St. Spec. 436 and 545 is to give standard procedure for
the design of involute straight spur, helical and bevel gears, resulting in maximum
efficiency and optimum contact conditions for each gear combination.

The time is past when the restriction offered by the necessity of interchangeable
change gears prevented individual treatment of each gear combination.

In modern machines each gear combination is mated for life because it was
realised that, besides being convenient, the running together of the same two gears
will improve performance.

This fact has been recognised in the B.S. mentioned above, and good contact
conditions can be ensured by applying a recommended method of correction to
pinion and gear addendum, dependent on the gear ratio and numbers of teeth.

It is high time that all engineers engaged in the design of machine members
realise that machine design procedures are gradually eliminating the trial and
error method. With advancement in engineering science and precision production
methods, our knowledge of actual conditions is daily increasing, hence the necessity
for a very high factor of ignorance is decreasing. Thus the Lewis formula disregards
the fact that all gears must have a contact ratio at least greater than unity, and,
therefore, the position of the point of application of load pressure when the full
load is carried by one pair of teeth is not at the tip but further down on the involute
profile. This also shows that pinion and gear are not equally strong in a combina-
tion if made of materials of equal strength.

Therefore, the Lewis formula must be regarded as too conservative and the
more correct approach, which forms the basis of the British Standard Specifications,
shall be accepted. This method, combined with values of stress factors amply
verified by practical results, gives a most satisfactory and versatile design pro-
cedure.

The object of this paper is to give an account of the fundamental principles on
which the British Standard Specifications for gear design are based, including a
detailed account of the method of correction and some recommendation in regard
to face width limitations.
THE INVLOUTE TOOTH PROFILE

The involute curve is developed from the base circle in the usual manner (See Fig. 2). The normal at any point \( P \) is obtained by drawing a line through \( P \), tangent to the base circle (See Fig. 3). The tangent of the involute is then obtained by drawing a line through \( P \) perpendicular to the normal. The radius of curvature is \( PP_1 \), where \( P \) is the point on the involute, and \( P_1 \), the instantaneous centre of curvature, is on the base circle (Fig. 3).

If two involute curves are in contact, the point of contact must be on the common normal and from the definition of the normal it follows that this must be tangent to both base circles. If the base circles rotate about their fixed centres \( O_p \) and \( O_w \) (see Fig. 4), the normal at any point of tangency must be the common tangent \( (T_pT_w) \) of the two base circles, hence all points of contact between the two involutes are on the common tangent which therefore is the path of contact. The pitch point \( (P) \) is obtained as the intersection of the path of contact \( (T_wT_p) \) and the line of centres \( (O_pO_w) \). \( P \) is the point where the pitch circles contact and

\[
\begin{align*}
r &= \frac{d}{2} = \frac{O_pP}{2} \\
R &= \frac{D}{2} = \frac{O_wF}{2}
\end{align*}
\]

are the pitch circle radii. It is obvious that the pitch circle radii will depend on the centre distance and on the base circle radii and that the same two involutes can work together at any centre distance. The pitch circle velocity is the same for both involutes and is in the direction of the pitch circle tangent at point \( P \). The pressure between the two involutes, however, is in the direction of the common normal (i.e. the line of action in case of involutes). Therefore the angle \( \psi \), being the acute angle between the line of action \((T_pT_w)\) and the pitch circle tangent at \( P \), is called the pressure angle.

The base circle radii \((r_o, R_o)\) and pitch circle radii \((r, R)\) are related by the following equations:

\[
\begin{align*}
r_o &= r \cos \phi \\
R_o &= R \cos \phi
\end{align*}
\]

and \( \phi \) will vary with the centre distance

\[
C = r + R.
\]

In standard practice 20° pressure angle is generally recommended but sometimes an increase in pressure angle up to 25° is necessary to ensure balanced tooth action and sufficient tooth strength.

The involute generated from a base circle with infinitely large radius is a straight line. The base circle degenerates also to a straight line, and the two lines are perpendicular to one another. This proves that the basic rack profile for involute gears is a straight line.

It can be seen from Fig. 5 that if a straight line is moving in such a manner that the involute with which it is in contact rotates and for a translation \( x \) in the direction \( V \) the angle of rotation \( \alpha \) of the involute is proportional to \( x \), i.e.

\[
\alpha = \frac{B_1C_1}{C_1} = \frac{QQ'}{r_o} = x \frac{\cos \psi}{r_o} = \text{constant} \cdot x
\]

the straight line will remain tangent to the involute. Hence the various positions of the straight line will envelop the involute curve. This is called generation and is the basis of involute gear generation from a basic rack cutter.
MODERN GEAR DESIGN PRACTICE.

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**FIG. 5. STRAIGHT SADDLED RACK GENERATES INVOLUTE**

- Pitch circle of pinion
- Base circle

**FIG. 6. EQUALLY SPACED INVOLUTES**

- Pitch circle of pinion
- Base circle

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**FIG. 3. TANGENT NORMAL RADIUS**

- \( r \), radius of curvature
- \( P_1 \), \( P_2 \)

**FIG. 4. TWO INVOLUTES IN CONTACT**

- Pitch circle of pinion
- Base circle

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**THE INVOLUTE CURVE**

- Arcc \( OP \)
- Arc \( O_2 \)

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**FIG. 2. CONSTRUCTION**

- \( P_1 \), \( P_2 \)
- Radii of curvature

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See also: [Further Reading](#)
DEFINITIONS AND BASIC PRINCIPLES RELATED TO STRAIGHT SPUR INVOLUTE GEARS

1. Definitions and notations.

The teeth on a gear are equally spaced and the measure of this spacing is called the *pitch*, in general.

The pitch can be given in various ways. For straight spur gears:

*Base pitch* \((p_o)\) is the spacing of consecutive involutes measured along the circumference of the base circle, hence if the number of teeth = \(t\)

\[
p_o = \frac{2r_o \pi}{t} = \frac{d_o \pi}{t} \quad \text{(Fig. 6)}
\]

*Circular pitch* \((p)\) is the measure of spacing along the circumference of the pitch circle. Hence (if \(d\) = diameter of pitch circle)

\[
p = \frac{d \pi}{t} \quad \text{(Fig. 6)}
\]

*Diametral pitch* = number of teeth to each inch of the pitch circle diameter:

\[
P = \frac{T}{D} = \frac{t}{d} \quad \text{i.e., } m = \frac{D}{T} = \frac{d}{t}
\]

Module = \(\frac{\text{pitch circle diameter}}{\text{number of teeth}}\)

Hence

\[
m = \frac{p}{\pi} = \frac{1}{P}
\]

In English countries the diametral pitch \((P)\) or circular pitch \((p)\) is mostly used, while in Europe the module \((m)\) is accepted as standard designation for tooth spacing. The portion of the involute which forms the gear tooth is between the addendum circle and the dedendum circle, thus defining the crest and root of the gear tooth.

Standard notations are as follows:

- \(R_o, r_o\) = base circle radii
- \(D_o, d_o\) = base circle diameters
- \(D, d\) = pitch circle diameters
- \(\psi\) = pressure angle
- \(J, j\) = addendum circle diameters
- \(I, i\) = dedendum circle diameters
- \(a = \frac{t-d}{2}\) and \(A = \frac{J-D}{2}\) is the addendum
- \(b = \frac{d-1}{2}\) and \(B = \frac{D-1}{2}\) is the dedendum
- \(B-a\) and \(b-A\) is the clearance (See Fig. 7)

2. Length of Contact, Contact Ratio, Active Tooth Profile.

The line of action is the straight line \(A_1B_1\) and contact between any two involute teeth takes place on this line. It is obvious that, since the tooth profile finishes on the addendum circle, no contact can take place beyond this circle. The *path of contact* therefore is that portion of the line of action which lies between the two addendum circles (i.e. between points \(A_2\) and \(B_2\), Fig. 7). The distance \(A_3B_3\) is the *length of path of contact*.

Furthermore, it is known that the distance between two consecutive involutes is equal to the base pitch \((p_o)\) (See Fig. 6). Therefore, the number of tooth-pairs...
in contact can be found by dividing the length of path of contact by \( p_0 \). This ratio is called contact-ratio \( (r_c) \).

\[
r_c = \frac{B_2A_2}{p_0}
\]

If this ratio is less than 1, the number of tooth-pairs in simultaneous contact is nought (0) or one (1), if the ratio is \( 1 < r_c < 2 \) the number of tooth pairs in contact is one (1) or two (2), etc. It is obvious that we must have at least one pair in contact all the time and therefore \( 1 < r_c < 2 \) at least.

As it can be seen from Fig. 7, the nearest point of contact to the pinion centre is \( A_2 \), and therefore the active tooth profile of the pinion is that portion of the tooth involute which lies between the addendum circle and a circle with radius \( = O_p A_2 \). Similarly the active tooth profile of the gear wheel is between the gearwheel addendum circle and a circle with radius \( O_p B_2 \).

3. **Maximum length of path of contact. Undercutting.**

No contact can take place beyond points \( A_1 \) and \( B_1 \) on the line of action hence distance \( A_2 B_1 \) is the maximum length of path of contact. At the same time, \( O_p A_1 \) and \( O_p B_1 \) are the maximum useful addendum circle radii (Fig. 7). If the addendum circle is extended beyond this maximum value, interference occurs and in order to eliminate this interference, the mating gear tooth must be undercut. This undercutting, however—

(a) weakens the tooth at its base;
(b) removes portion of the active tooth profile and therefore shortens the path of contact and diminishes the contact ratio.

Undercutting must be avoided if favourable contact conditions and strong gear teeth are desired.

4. **Methods to eliminate undercutting.**

Since all gears are generated from the basic rack, conditions of interference and undercutting between basic rack cutter and gear or pinion must be investigated. It is sufficient to investigate gears generated from a basic rack cutter with \( p = \pi \) (i.e. unit diametral pitch \( P = 1 \)), because all gear combinations which differ only in pitch are geometrically similar.

From Fig. 8 it can be seen that as long as the distance between centre \( O_p \) and the line of symmetry of the basic rack \( L \geq A + r_o \cos \psi \) no portion of the active tooth profile will be removed. However, weakening of the root of the gear tooth will still occur as long as \( L < r_o + A \), i.e. as long as the root circle of the gear is below the base circle. Instead of the root circle a circle with radius \( \left( \frac{d}{2} - a \right) \) can be considered since the influence of the clearance can be neglected on account of the usual radius at the root.

As shown in Fig. 8, undercutting can be eliminated—

(a) by increasing the pressure angle from \( \psi \) to \( \psi_1 \);
(b) by reducing the tooth depth from \( W \) to \( W_1 \);
(c) by correction (Fig. 9).

It is obvious that methods (a) and (b) would necessitate different rack cutters for different gear combinations if effectively done.

In standard practice, method (a) is only used in extreme cases, while a compromise solution for method (b) is the stub tooth system used in the U.S.A. The recommended British Standard practice, however, is method (c) which leaves the proportions of the basic rack cutter unchanged (constant tooth depth and pressure angle) but varies the addendum and dedendum of pinion and gear wheel. The method of correction will be explained in detail as follows.

If the condition shown in Fig. 8 arises, i.e.,

\[
L < A + r_o \cos \psi \text{ or } A + r_o \cos \psi < L < r_o + A
\]

undercutting can be wholly corrected or conditions improved by withdrawing the rack cutter, thus increasing the distance \( L \). The amount by which the rack has been withdrawn is called the correction \( (K) \) (Fig. 9) and is added on to the standard
If the correction is divided by the uncorrected addendum we obtain the correction coefficient

\[ k = \frac{K}{A} \]

This coefficient is the same for all geometrically similar combinations.

For a gear with \( p = \pi \) the pitch diameter \( d = \frac{t_p}{\pi} = t \) (number of teeth). From Fig. 10 the number of teeth in gears corresponding to \( OP = L_o = A + r_o \cos \phi \) and \( MP = L_m = A + r_o \) can be calculated \( (t_o \) and \( t_m) \).

1. \( OP = \frac{t_o}{2} = A + \frac{t_o}{2} \cos^2 \phi \)

\[ t_o = \frac{2A}{1 - \cos \phi} = 2A \cosec \phi \]

2. \( MP = \frac{t_m}{2} = A + \frac{t_m}{2} \cos \phi \)

\[ t_m = \frac{2A}{1 - \cos \phi} \]

In standard gear practice the uncorrected addendum \( a = A = m = \frac{p}{\pi} = \frac{1}{p} \).

The whole depth = \( 2 \cdot 25m = \frac{2 \cdot 25p}{\pi} = \frac{2 \cdot 25}{p} \) resulting in a clearance

\[ c = 0 \cdot 25m = \frac{0 \cdot 25p}{\pi} = \frac{0 \cdot 25}{p} \]

Exceptions are precision ground gears where more clearance is needed. For \( p = \pi \), i.e. \( P = 1 \), and \( A = \frac{1}{P} = 1 \), thus \( t_o = 2 \cosec \phi \) and \( t_m = \frac{2}{1 - \cos \phi} \).

If \( \phi = 20^\circ \) we obtain \( t_o = 17 \), \( t_m = 30 \) and if \( \phi = 14 \frac{1}{2}^\circ \) we obtain \( t_o = 32 \)

If \( t < t_o \) and we wish to avoid cutting away the active profile, the minimum correction to be applied is distance \( QQ' \) (Fig. 10). But

\[ K_o = QQ' = PQ - PQ' = \frac{t_o}{2}(1 - \cos^2 \phi) - \frac{t}{2}(1 - \cos^2 \phi) \]

and the correction coefficient

\[ k_o = \frac{K_o}{A} = 1 - \frac{t}{t_o} = \frac{1}{t_o}(t_o - t) \]

If all weakening of root is to be eliminated \( (L = A + r_o) \)

\[ k_m = \frac{K_m}{A} = \left( 1 - \frac{t}{t_m} \right) = \frac{1}{t_m}(t_m - t) \]

For \( \phi = 20^\circ \) we have:

\[ k_o = \frac{1}{t_o}(t_o - t) = \frac{1}{17}(17 - t) = 0 \cdot 0588(17 - t) \]

\[ k_m = \frac{1}{t_m}(t_m - t) = \frac{1}{30}(30 - t) = 0 \cdot 033(30 - t) \]

For effective correction the minimum correction coefficient \( k_p > k_o \) and if possible \( k_p > k_m \).
5. The involute function and the effect of correction on pressure angle and centre distance.

If $O_0P_o$ is selected as co-ordinate axis (Fig. 11), a point $P_1$ on the involute curve starting from $P_o$ on the base circle will be determined by its polar co-ordinates $r_1$ and $\delta$ where $\delta$ is the angle between the direction $r$ and $O_0P_o$.

If angle $\angle P_1'OP_1 = \phi$, the following equations can be written:

$$ r = r_o \sec \phi $$

(1)

and

$$ \tan \phi = \frac{P_1P_1'}{r_o} $$

(1a)

but $P_1P_1' = \text{arc. } P_oP_1' = r_o(\phi + \delta)$

hence

$$ \phi + \delta = \frac{P_1P_1'}{r_o} = \tan \phi \text{ from which} $$

$$ \delta = \tan \phi - \phi = \text{inv} \phi \quad (2) $$

This latter is called the involute function of $\phi$ and (1) and (2) give the polar co-ordinates of point $P_1$.

Values of $\text{inv} \phi$ can be obtained from tables of involute functions [See Bibliography (3)]. Considering now the following problem (Fig. 12), “given two involutes, each generated from the same base circle (radius $r_o$), find the distance of the two involutes if one point on each involute is given as indicated on Fig. 12.” The two involutes are “$g$” and “$e$” and the given points on each are $P_1$ on “$g$” and $Q_2$ on “$e$”.

The distance of the two involutes is equal to $P_1Q_2 = P_1Q_1 = P_2Q_2$ arc. But arc $P_0Q_2 = r_o \phi$. $\phi$ = angle $P_0Q_0 = \delta_2 - \delta_1$ but from previous definitions

$$ \delta_1 = \text{inv} \phi_g $$

$$ \delta_2 = \text{inv} \phi_e $$

and thus

$$ P_1Q_1 = r_o \phi = r_o (\text{inv} \phi_1 - \text{inv} \phi_e) \quad (3) $$

If involute “$g$” represents the tooth profile of an involute gear with no correction and “$e$” the tooth profile after correction, generated from a straight-sided rack with $\phi = \phi_g$ and $P = 1$, then (Fig. 13) the correction $k$ necessary to obtain this result can be calculated as follows:

$$ k = P_1Q_1 \csc \phi_g = r_o \csc \phi_g (\text{inv} \phi_1 - \text{inv} \phi_e) \quad (4) $$

If for a gear of unit diametral pitch the correction is equal to $k = P_1E$, the basic involute will be displaced from position $g$ to position $e$. Involute $e$ will cut line $OP_1$ in point $Q_2$. From (3) and (4)

$$ P_1Q_1 = \frac{t}{2} \cos \phi_g (\text{inv} \phi_1 - \text{inv} \phi_e) = k \sin \phi_g $$

hence

$$ \text{inv} \phi_1 = \frac{2k \tan \phi_g}{t} + \text{inv} \phi_e $$

Taking now two gears, a pinion and a gear wheel, and applying $k_p$ resp $k_w$ correction, we obtain angles $\phi_1$ (for pinion) and $\phi_2$ (for gear wheel) from

$$ \text{inv} \phi_1 = \frac{2k_p \tan \phi_g}{t} + \text{inv} \phi_g \quad (5a) $$

and

$$ \text{inv} \phi_2 = \frac{2k_w \tan \phi_g}{T} + \text{inv} \phi_g \quad (5b) $$

If now the two gears are brought in contact, the point of contact, denoted by $Q_2$, must be on the common tangent of the base circles (See Fig. 14). It is obvious from Fig. 14 that in general the pressure angle $\psi_k$ at which the two gears will mesh, is not the same as the pressure angle of generation ($\phi_k$). This new pressure angle however can be calculated as follows (from Fig. 14):—
MODERN GEAR DESIGN PRACTICE.

\[ Q_a P = \frac{T}{2} \cos \psi_e (\text{inv} \psi_z - \text{inv} \psi_e) \]

\[ = \frac{t}{2} \cos \psi_e (\text{inv} \psi_e - \text{inv} \psi_z) \]

\[ \text{i.e. } t \text{ inv} \psi_1 + T \text{ inv} \psi_2 = (t + T) \text{ inv} \psi_e \]

(6)

Using (5a) and (5b)—

\[ t \text{ inv} \psi_1 = 2k_p \tan \psi_g + t \text{ inv} \psi_e \]

\[ T \text{ inv} \psi_2 = 2k_w \tan \psi_g + T \text{ inv} \psi_e \]

by addition we obtain:

\[ t \text{ inv} \psi_1 + T \text{ inv} \psi_2 = 2(k_p + k_w) \tan \psi_g + (t + T) \text{ inv} \psi_e \]

(7)

\[ \text{inv} \psi_e = \frac{2(k_p + k_w) \tan \psi_g + (t + T) \text{ inv} \psi_e}{t + T} \]

Inv \psi_e can be calculated from above equation and \( \psi_e \) obtained from tables of involute functions.

The two gears will engage under pressure angle \( \psi_e \).

If \( k_p + k_w = 0 \), \( \psi_e = \psi_g \) and centre distance \( C = \frac{1}{2}(t + T) \); if, however,

\( k_p + k_w > 0 \), \( \psi_e > \psi_g \) and the centre distance will increase to

\[ C = \frac{1}{2}(t + T) \cos \psi_e \sec \psi_e \]

(Fig. 15)

This means that the pitch diameters and also the circular pitch are increased as follows:

\[ d_e = t \cos \psi_e \sec \psi_e \]

\[ D_e = T \cos \psi_e \sec \psi_e \]

and the diametral pitch \( P_e = \frac{t}{d_e} = \sec \psi_e \cos \psi_e \).

The base pitch, however, remains unchanged and is determined by the generating rack.

The increased pressure angle of engagement (\( \psi_e \)) and the necessary increase in centre distance can be calculated if \( k_p + k_w \) is known.

Knowing the effect of correction on the centre distance of mating gears, we can now investigate the two distinct cases.

Case 1.

If \( k_p + k_w = 0 \), i.e. \( k_w = -k_p \)

It was shown that for \( t \) number of teeth in the pinion preferably a minimum correction coefficient \( k_p = k_m = \frac{1}{t_m} \) \((t_m - t)\) shall be used. If this is done, a negative correction coefficient \( k_w = -k_p \) must be applied to the wheel in order to satisfy the above condition of \( k_p + k_w = 0 \). This can be safely done as long as

\[ \frac{1}{t_m} (t_m - T) \geq \frac{1}{t_m} (t_m - t) \]

\[ \text{i.e., } t + T \geq 2t_m \]

for \( \psi_g = 20^\circ \), \( t_m = 30 \), hence the condition \( k_p + k_w = 0 \) can be used whenever

\[ t + T \geq 60 \]

In cases where \( \frac{1}{t_m} (t_m - T) > \frac{1}{t_m} (t_m - t) \) the correction for the pinion can be increased beyond the value given by \( k_m \).

This means that if we denote the correction coefficient at which the addendum line of the generating rack is tangent to the base circle by \( k_{m_p} \) (for the pinion) and by \( k_{m_w} \) (for the gear wheel), then for cases where \( -k_{m_w} > k_{m_p} \) the correction coefficient \( k_p \) can be greater than \( k_{m_p} \).

For \( \psi_g = 20^\circ \) this condition is reached when \( t = 17 \) and \( T > 43 \),

\[ k_{m_p} = \frac{1}{30} = \frac{13}{30} \text{ and } -k_{m_w} > \frac{43}{30} - 1 = \frac{13}{30} \]
and \( k_p \) can be increased beyond \( k_{m_p} \) by an amount corresponding to one half of the excess negative correction permissible for the gear wheel. Thus we obtain

\[
k_p = k_{m_p} + \frac{1}{2}(-k_{m_w} - k_{m_p})
\]

\[
2k_p = 2k_{m_p} - k_{m_w} - k_{m_p} = k_{m_p} - k_{m_w}
\]

\[
k_p = \frac{1 - \frac{1}{t_m}}{2} - \left(1 - \frac{T}{t_m}\right) = \frac{T - t}{2t_m}
\]

or

\[
k_p = \frac{T}{2t_m} \left(1 - \frac{1}{T}\right)
\]

for \( \psi = 20^\circ, 2t_m = 60 \), and

\[
k_p = \frac{T}{60} \left(1 - \frac{1}{T}\right) = -k_{m_w}
\]

However, as will be shown later on, the aim of correction is not only to eliminate undercutting but also to balance the pinion and gear wheel strength and to improve sliding contact conditions with the view of decreasing wear. To achieve this aim a compromise procedure has been adopted for correction in the B.S. Specification 436. This compromise is essentially an empirical modification of the above principle, substantiated by the results obtained from actual gear combinations.

Case 2.

If \( t + T < 2t_m \) the condition \( k_p + k_w = 0 \) cannot be fulfilled without seriously weakening the gear wheel.

In such cases the correction applied to the pinion is greater than the negative correction applied to the wheel, i.e. \( k_p + k_w > 0 \) and the centre distance must be extended. The relation between \( k_p + k_w \), the increased pressure angle \( \psi_e \) and the necessary extension of centre distance (\( \Delta \)) for \( p = \pi (m = \frac{1}{P} = 1) \) can be obtained by referring to Figures 14 and 15 and are given as follows:

The pressure angles \( \psi_e \) at which the corrected gears will mesh without backlash can be calculated from

\[
\text{inv} \psi_e = \frac{2(k_p + k_w)}{t + T} \tan \psi_e + \text{inv} \psi_e
\]

and the extended centre distance (Fig. 15) from

\[
C_e = (R_e + r_0) \sec \psi_e = \frac{1}{2} \left[ (T + t) \cos \psi_e \sec \psi_e \right]
\]

from which the extension [with \( C = \frac{1}{2}(t + T) \)]

\[
\Delta = C_e - C = \frac{1}{2} \left[ (T + t) \cos \psi_e \sec \psi_e - 1 \right]
\]

6. Effect of correction on strength and specific sliding.

It is obvious from the above that a positive correction will increase the tooth strength and give a tooth form with a more uniform strength (Fig. 16).

It can also be easily understood, that the larger the number of teeth, the stronger the root of the tooth will be, all other things being equal (Fig. 17), because the radius of curvature of the involute at the point \( P \) is increasing with increasing base circle (Fig. 17B).

This means that without correction, the wheel tooth has a stronger root than the pinion tooth. (The relative strength, however, will depend on the position of the point of application of the maximum tooth pressure.) If the number of teeth in the wheel is very much greater than in the pinion, this may mean that the pinion is much weaker if material of the same strength is used for both gears and therefore the capacity of the gear combination is determined by the weaker member (the pinion). By applying equal but opposite correction, this difference in strength can be somewhat balanced.

The load capacity and life of a gear combination are, however, influenced not only by the maximum stress but also by the surface stress at the point of contact and by the velocity of sliding.
It will be shown in the following discussion that surface contact conditions can also be influenced by correction.

The conditions at the point of contact between two involute surfaces can be compared to those existing between two cylinders in contact having radii equal to the radius of curvature of the two involutes at the point of contact and which roll and slide with the same velocity as the involute surfaces do at this point.

It is known that the surface stress is increasing with the pressure between the two surfaces and is decreasing with increasing relative radius of curvature ($R_r$) where

$$R_r = \frac{R_1 R_2}{R_1 + R_2}$$

$R_1$ and $R_2$ being the instantaneous radii of curvature of the surfaces. For involute gears in contact $R_1 + R_2 = \text{const.} = H$.

Taking unit diametral pitch, $H = \frac{1}{2}(t + T) \sin \psi$ (Fig. 18),

$$R_2 = H - R_1 \quad \text{and} \quad R_r = \frac{(H-R_1)R_1}{H}$$

The maximum value of $R_r$ will be at a point for which

$$\frac{dR_r}{dR_1} = \frac{1}{H} (H - 2R_1) = 0,$$

i.e., $R_1 = \frac{H}{2}$, $R_2 = \frac{H}{2}$ and $R_r = \frac{H}{4}$

in general, $R_r$ decreases as the difference $R_1 - R_2$ increases.

While the pressure is determined by the gear load, the relative radius of curvature and the rate of sliding can be influenced by suitable design. If the involute of the pinion rotates with an angular velocity $\omega_1$, the involute arc corresponding to a time interval $d\tau$ can be obtained as follows (See Fig. 19). The involute arc for the pinion will be $R_1 \omega_1 d\tau$, and the corresponding arc of the gear involute in contact for the same time interval is $R_2 \omega_2 d\tau$ where $\omega_2 = \text{angular velocity of gear involute}$.

$R_1$ and $R_2$ are the instantaneous radii of curvature for the pinion and gear at the point of contact.

The amount of sliding will be

$$(R_1 \omega_1 - R_2 \omega_2) d\tau$$

with respect to the pinion

and

$$(R_2 \omega_2 - R_1 \omega_1) d\tau$$

with respect to the gear.

Hence the rate of sliding for the pinion

$$\frac{R_1 \omega_1 - R_2 \omega_2}{R_1 \omega_1} = 1 - \frac{R_2 \omega_2}{R_1 \omega_1} = 1 - \frac{R_2}{R_1} \frac{t}{T}$$

and for the gear

$$\frac{R_2 \omega_2 - R_1 \omega_1}{R_2 \omega_2} = 1 - \frac{R_1 \omega_1}{R_2 \omega_2} = 1 - \frac{R_1}{R_2} \frac{T}{t}$$

At the pitch point $R_1 = \frac{d}{2} \sin \psi$ and $R_2 = \frac{D}{2} \sin \psi$,

hence $\frac{R_1}{R_2} = \frac{d}{D} = \frac{t}{T}$ and thus the rate of sliding will become zero. During contact the rate of sliding will vary from a negative maximum to zero at the pitch point, increasing to a positive maximum or vice versa.

If favourable contact conditions are required, the absolute value of the negative maximum shall not be very far from the positive maximum and the change in rate shall not be very different for the pinion and gear wheel.

In the following example it will be shown how, by correction, conditions for wear can be improved (Fig. 20), i.e., how the ratio $\frac{R_1}{R_3}$ and the relative radius of curvature $R_r$ can be influenced. Contact starts at point $C_1$, the corresponding radii of curvature being $R_1'$ and $R_2'$ and contact ceases at point $C_2$, the radii of curvature...
at this point being $R_1'$ and $R_2'$. From Fig. 20 we have (for unit diametral pitch):

$$R_1' = \frac{t}{2} \sin \psi - lw_1$$

$$R_2' = \frac{T}{2} \sin \psi + lw_1$$

$$R_1'' = \frac{t}{2} \sin \psi + \phi_1$$

$$R_2'' = \frac{T}{2} \sin \psi - \phi_1$$

Taking a numerical example—

$t = 20, \ T = 60, \ \psi = 20^\circ$, we obtain

$$\frac{t}{2} \sin \psi = 10. \sin 20^\circ = 3.4202$$

$$\frac{T}{2} \sin \psi = 30. \sin 20^\circ = 10.2606$$

If we apply correction such as $k_p = -k_w$ the tabulated results are as follows. See Table 1.)

<table>
<thead>
<tr>
<th>$k_p = -k_w$</th>
<th>$R_1'$</th>
<th>$R_2'$</th>
<th>$R_1''$</th>
<th>$R_2''$</th>
<th>$l_{p1}$</th>
<th>$l_{w1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7838</td>
<td>12.8970</td>
<td>5.7152</td>
<td>7.9656</td>
<td>2.6364</td>
<td>2.2950</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2602</td>
<td>12.4206</td>
<td>6.1802</td>
<td>7.5006</td>
<td>2.16</td>
<td>2.76</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5302</td>
<td>12.1506</td>
<td>6.2602</td>
<td>7.4206</td>
<td>1.89</td>
<td>2.84</td>
</tr>
</tbody>
</table>

The figures are self-explanatory. With no correction the rate of sliding on the pinion is more than 3 times that on the gear, but with increasing correction this disparity gradually disappears without unduly increasing the rate of sliding on the gear wheel. At $k_p = 0.3$ the rate of sliding is almost equal for pinion and gear. The length of contact decreases slightly but the contact ratio is still well above unity. An increase of correction beyond 0.3 would mean no further benefit.

Summing up above discussions, it can be stated that by using correction in a suitably selected manner, the following improvements can be achieved:

(a) elimination or decrease of undercutting in pinion thus giving the tooth a form of greater strength;

(b) balancing to a certain extent relative strength of pinion and gear wheel;

(c) obtaining more favourable surface stress, and sliding contact conditions thus decreasing the wear of the gear combination.

The British Standards Institution recommends $\psi = 20^\circ$ standard pressure angle.
constant whole depth \( 2.25m = \frac{2.25}{P} = \frac{2.25p}{\pi} \) for all cases except precision ground gears.

For equal number of teeth when \( t > 30 \) and \( T > 30 \), the basic addendum
\[
\frac{1}{P} = \frac{p}{\pi} = m.
\]

For unequal number of teeth the following method of correction is recommended.

*For Straight spur gears.*

Corrected addendum of pinion \( a = \frac{1}{P} (1 + k_p) \)

... " wheel \( A = \frac{1}{P} (1 + k_w) \)

\( k_p \) and \( k_w \) are the actual corrections for \( P = 1 \) and are obtained in the following way.

**Case 1.**

\[
t + T \geq 60 \quad \text{and} \quad t \neq T
\]

\[
k_p = 0.4 \left(1 - \frac{t}{T}\right) \text{ or } 0.02(30 - t) \text{ whichever is greater}
\]

\[
k_p + k_w = 0 \quad k_w = -k_p
\]

**Case 2.**

\[
t + T < 60
\]

\[
k_p = 0.02(30 - t) \quad \text{and} \quad k_w = 0.02(30 - T)
\]

\[
k_p + k_w = 0.02[60 - (t + T)] > 0
\]

The centre distance must be extended. The increased pressure angle and increased centre distance can be calculated as previously outlined in (5). By extending the centre distance, the ratio remains unchanged but the pitch diameters increase, hence the circular pitch also increases slightly.

The outside diameters are obtained for both case 1 and 2, by adding twice the addendum to the generated pitch diameter, hence:

\[
j = d + 2a = \frac{t}{P} + \frac{2}{P} (1 + k_p) = \frac{1}{P} (t + 2 + 2k_p)
\]

\[
J = D + 2A = \frac{T}{P} + \frac{2}{P} (1 + k_w) = \frac{1}{P} (T + 2 + 2k_w)
\]

While the addendum is corrected, the dedendum changes also, but the clearance remains constant as long as the centre distance is not extended.

For a pinion engaging a rack \( (T = \infty) \) the correction \( k_p = 0.4 \left(1 - \frac{t}{T}\right) = 0.4 \) and for internal gears which give a better contact ratio than rack the B.S. 436 recommends the same correction as for rack, i.e.

\[
k_p = 0.4 \quad k_w = -k_p = -0.4
\]

From the original analysis of correction, it can be seen that a method of correction can be worked out for any pressure angle. A similar method to above only with different constants can be used for 14\( \frac{2}{3} \)° pressure angle.

**Example.**

Taking again the same example, \( t = 20, T = 60, \psi = 20^\circ \), the recommended correction is \( 0.4 \left(1 - \frac{t}{T}\right) = 0.4 \left(1 - \frac{20}{60}\right) = 0.265 \). This value is not far from the value 0.3 obtained previously, giving balanced sliding conditions.
MODERN GEAR DESIGN PRACTICE

PART 3

COMPARATIVE STRENGTH AND DURABILITY OF STRAIGHT SPUR GEARS

1. Introduction.

The problem of finding the stresses and wear in a gear tooth, under actual working condition, is a very complex one. However, this does not mean that the problem cannot be tackled in a logical and scientific manner using stress values which are determined from data obtained by practical experience.

The method used, to obtain comparison between different gear combinations, assumes all conditions, the analytical treatment of which would be difficult, constant and analyses only the effect of variables which can be treated analytically with ease.

In the calculations to follow, we will assume—
(a) perfect tooth form;
(b) static conditions of tooth engagement;
(c) uniform tooth pressure along the gear tooth;
(d) contact ratio $r_c$ between one and two.

The deviation from assumption (a) can be controlled by the manufacturing process and for a given method of production will be at the same rate for all gear combinations.

The deviation from static conditions (assumption (b)), can be accounted for by an appropriate adjustment of the stress factors dependent on speed and service conditions. (This procedure assumes however that the increment load resulting from dynamic conditions is proportional to the load.) Assumption (c) is not correct, but the deviation from it can be made negligible if the tooth length is suitably determined. (This problem was analytically treated by H. Poritsky, A. D. Sutton and A. Pernick, Journal of Applied Mechanics, June 1945, page A-78.)

Assumption (d) is correct for a large number of gear combinations in everyday use.

2. Stresses in a gear tooth.

If as assumed, the contact ratio is between one and two, the number of tooth pairs in contact varies between one and two. As long as two pairs are in contact, the load is divided and carried by two teeth. This condition however is suddenly changed when one of the two pairs leaves contact and in consequence the full load is carried by one pair. Two such positions exist for each pair of gears. Figs. 21A and 21B show those two positions.

The position shown in Fig. 21A represents the worst stress condition for the pinion while Fig. 21B shows the position in which the stresses in the gear wheel tooth will be maximum. It is also clear that for a contact ratio greater than unity, the point of application of the full tooth pressure is not at the tip of the tooth as assumed in Lewis's formula.

The point of contact representing maximum stress conditions can be easily determined for any gear combination.

Fig. 22 shows one tooth of the pinion in the position represented by Fig. 21A and we can proceed with the calculation of the maximum stress.

The following notations will be used:
- $F_t$ = tangential force per unit face width (lb./inch).
- $F_n$ = normal tooth pressure per unit face width (lb./inch).
- $\psi$ = pressure angle.
- $\theta$ = acute angle between line of action and a line perpendicular to the axis of symmetry of the gear tooth in which the stress is calculated.
Considering unit face width we obtain—

\[ F_n = F_t \sec \psi \]  

\((1)\)

This force \((F_n)\) is in the direction of the line of action and the point of application is \(C_4\) on the tooth profile. If the intersection of the line of action with the axis of symmetry of the tooth is denoted by \(B\), \(F_n\) can be considered as acting at this point.

Resolving it into two components in the direction of the axis of symmetry \(OB\) and in a direction perpendicular to this \((LL)\) we obtain—

\[ F_n \sin \theta \text{ (in direction } OB\text{)} \]

and

\[ F_n \cos \theta \text{ (in direction } LL\text{)} \]

The tooth will be compressed by \(F_n \sin \theta\) and \(F_n \cos \theta\) will cause bending and shear.

The tooth thickness at the root can be found by drawing a parabola with vertex at \(B\) and tangent to the root radii of the tooth at points \(G\) and \(H\). (Point \(G\) can be found by trial considering that the tangent of the parabola at point \(G\) must be halved by line \(LL\), i.e. \(GJ = JK\).) The bending moment at the root section will be \(x.F_n \cos \theta\), where \(x\) is the distance of line \(LL\) from \(GH\). There will be stress concentration at the root radius. Neglecting the effect of shearing forces, we obtain that the maximum stresses will be at point \(G\) (compressive).* (See note and ref. 5 in Bibliography.)

Denoting—

\[ n_c = \text{direct compressive stress at point } G \]

\[ n_b = \text{bending stress at point } G \]

\[ k_c = \text{stress concentration factor in compression} \]

\[ k_b = \text{stress concentration factor in bending} \]

we obtain for unit face width

\[ n_c = \frac{F_n \sin \theta}{1.y} = \frac{F_t \sec \psi \sin \theta}{y} \]

\[ n_b = \frac{x.F_n \cos \theta}{1.y^2} = \frac{6x.F_t \sec \psi \cos \theta}{y^2} \]

and the permissible average stress for the purpose of calculations (named the bending stress factor) is the sum of above quantities, i.e.

\[ S_b = \frac{n_c}{k_c} + \frac{n_b}{k_b} = \frac{F_t \sec \psi \left(\sin \theta + \frac{3x \cos \theta}{y^2}\right)}{y^2 \cos \phi} = \frac{F_t}{y^2 \cos \phi} \]

The denominator represents a length which is proportional to the pitch for geometrically similar gear combinations. It is therefore sufficient to find its value for unit diametral pitch. This value (calculated for unit diametral pitch) is called the strength factor and is denoted by \(Y\).

\[ Y = \frac{y_1^2 \cos \phi}{y_1 \sin \theta + 6x_1 \cos \theta} \]

\((y = y_1 \text{ for } P = 1)\)

and since in general \(y = \frac{y_1}{P}\) \(x = \frac{x_1}{P}\)

we obtain

\[ \frac{y^2 \cos \phi}{y \sin \theta + 6x \cos \theta} = \frac{Y}{P} \]

The permissible tangential force per unit face width will be for static conditions

\[ F_t = \frac{S_b \cdot Y}{P} \]

* Note.—Recent investigations did show that if a failure is due to excessive bending stresses and not to surface stresses, the fatigue crack will start at the tension fillet, and not at the compression fillet, where the stress is maximum.
MODERN GEAR DESIGN PRACTICE

Fig. 21. Position for max. surface stress.

Fig. 22. Derivation of strength factor.

Fig. 21A. Max. stress in pinion tooth.

Fig. 21B. Max. stress in gearwheel tooth.
However, to allow for effects resulting from motion, such as inertia and impact due to error in tooth action, variation in load, effect of speed and running time, etc., a speed factor $X_b$ must be introduced.*

To find the total permissible tangential force ($F$) we have to take into account the face width ($f$). Thus we have for uniform load distribution along the tooth length (face width)—

$$F = fF_t = X_b S_b f \frac{Y}{P}$$

The speed factor $X_b$ is dependent on the number of revolutions per minute and regular variations in speed and loading within a day. It accounts for fatigue and inertia effects (dynamic load) for average gear drives manufactured with an accuracy commensurate with speed. The safe horsepower capacity can be calculated as follows:

$$\text{Horsepower for strength} = \frac{d}{2} \frac{2\pi n}{33,000} = \frac{X_b S_b f Y n t}{120,000 P^2}$$

where

- $X_b = \text{speed and running time factor for strength (dimensionless)}$
- $S_b = \text{bending stress factor of the gear material (p.s.i.)}$
- $f = \text{tooth length (face width) (inch)}$
- $Y = \text{strength factor (inch)}$
- $n = \text{revolutions per minute}$
- $t = \text{number of teeth}$
- $P = \text{diametral pitch}$

### 3. Factors influencing wear.

While the actual stress will account for sudden failure, the useful life of a gear combination will mainly depend on conditions of contact. The problem of wear prevention is much more complex than the previously discussed strength criterion. Hence a rigid treatment of this problem is impractical.

While actual breakage due to overstressing very rarely occurs in properly designed, accurately manufactured and assembled gear combinations, certain types of surface deterioration may occur more often. The following are the types which can be more or less influenced by correct design procedure:

- (a) Wear due to boundary friction.
- (b) Wear due to dry friction (scoring and galling).
- (c) Pitting.
- (d) Burning.
- (e) Rolling and peening.

Gear tooth contact is essentially a rolling-sliding contact between two curved surfaces which are pressed against one another by a periodically varying pressure. The surface, therefore, is subjected to compression on very small areas of contact. It is known that this type of loading often will cause maximum stress below the surface and it is believed that if, due to excessive local pressure, this stress exceeds the fatigue limit of the material, a fatigue crack will develop and pitting will occur. Two types of pitting are known. Initial pitting which is caused by high surface compression per unit area due to insufficient area of contact. As normal wear advances, and this area of contact increases, surface compression will drop below the critical value and pitting will stop and disappear. If proper boundary lubrication is secured, normal wear will also reach an equilibrium condition because the pressure and thus the friction force decreases. This procedure is called running in. If, however, lubrication is insufficient, scoring and galling may occur.

Progressive pitting occurs when the surface compression exceeds a certain value and the pit formation is more rapid than the increase in area due to normal wear. If this occurs, the actual area of contact decreases all the time, thus causing a gradual increase of the surface stress. The number of small craters will increase as the

* **Note.**—The effect of inertia and impact due to error will be discussed later in more detail.
fatigue cracks become more numerous and finally complete destruction of the tooth surface takes place.

_Burning_ is caused by excessive heat being generated by friction. High speed combined with high pressure will cause a high temperature which will destroy the mechanical properties of the surface layer, and breakdown will result. Very often in such cases the lubricating power of oil is also destroyed resulting in welding and seizure.

_Rolling and peening_ is due to the squeezing out of material caused by the rolling-sliding action and usually occurs when the pressure is excessive and the materials in contact are too soft and ductile. It can be recognised easily by the formation of fins at the tip and along the contour.

It can be seen from the foregoing that surface failure can be influenced and avoided by—

(a) proper selection of materials in contact (to avoid excessive boundary friction and rolling and peening);
(b) proper selection of lubricant (using E.P. lubricants when high pressure cannot be avoided);
(c) limiting surface loading to safe value (to stabilise wear and pitting);
(d) applying speed factor \(X_c\), dependent on speed and service conditions (to allow for fatigue, and prevent breakdown due to excessive temperature).

The limiting of surface loading is done in the following way:

In present practice the following relation is used:

\[
S_c = \frac{F_n}{(R_1/R_2)^{0.8} \cos \psi} = \frac{F_t}{(R_1/R_2)^{0.8}} \cdot \cos \psi
\]

\(S_c\) = surface stress factor for static conditions and low speed determined experimentally for various gear materials.

\(F_n\) = normal tooth pressure per unit length of tooth (lb. per inch).

\(F_t\) = tangential force per unit length of tooth (lb. per inch).

\(R_r = \frac{R_1 R_2}{R_1 + R_2}\) = relative radius of curvature (inch).

\(\psi\) = pressure angle.

\(f\) = tooth length (inch).

The permissible total tangential force for wear can be calculated by applying the _speed factor_ \(X_c\). Thus:

\[
F = X_c f F_t = X_c S_c f (R_1/R_2)^{0.8} \cos \psi
\]

For geometrically similar combinations the relative radius of curvature is proportional to the pitch. It is sufficient therefore to calculate the relative radius of curvature for unit diametral pitch at the point of contact where the load is maximum and the relative radius is minimum.

Referring to Fig. 23, point \(C_4\) will be the required point because the full load will be carried by one tooth pair from point \(C_3\) to \(C_4\), but the relative radius of curvature is smallest at the point nearer to the centre of the pinion. (See also Fig. 20 and Table 1 in numerical example.) Calculating for unit diametral pitch, the value of \((R_1/R_2)^{0.8} \cos \psi = Z\) where \(Z\) is called the _zone factor_. For any other diametral pitch

\[
R_r = \frac{R_1}{P}
\]

hence the value

\[
(R_1/R_2)^{0.8} \cos \psi = \left(\frac{R_1}{P}\right)^{0.8} \cos \psi = \frac{Z}{P^{0.8}}
\]

and

\[
F = X_c S_c f F_t = X_c S_c f Z P^{0.8}
\]

_The safe horsepower capacity for wear can be calculated in the same manner as for strength._

\[
\text{Horsepower for wear} = \frac{f d}{2} \times \frac{2 \pi n}{33,000.12} = \frac{X_c S_c f Z P^{0.8} n}{126,000 P^{0.8}}
\]
4. Influence of tooth length on distribution of tooth pressure along the tooth of the pinion.

In the above calculations it was assumed that the load is uniform along the tooth. It can be shown, however, by mathematical analysis (Paper on "Distribution of Tooth Load along a Pinion," by H. Poritsky, A. D. Sutton and A. Pernick, Journal of Applied Mechanics, June 1945, page A-78), that assuming perfect tooth profile, rigid gear shaft, ribs and rim, but a flexible pinion, furthermore neglecting the effect of journal clearance and oil film on deflection, a high concentration of load occurs at the load end of the pinion tooth while in the middle the load is well below average for a long pinion.

The analysis is based on proportionality between load and deflection. The variation of tooth deflection (δ) along the pinion tooth is found by solving the following differential equation:

\[ \frac{d^2\delta}{dx^2} + \frac{a}{EJ} \delta = 0 \]

where \( a = \frac{d^2fS}{4GJ} \) and \( \beta = \frac{Sf^4}{EI} \).

and

\[ d = \text{pitch diameter of pinion} \]

\( f = \text{tooth length (face width)} \)

\( S = \text{combined tooth stiffness of gear and pinion transferred to the pinion (lb per inch face width per inch deflection)} \)

\( E \) and \( G \) are moduli of elasticity in tension and torsion

\( J = \text{polar moment of inertia of the cross-section of the pinion} \)

\( I = \text{moment of inertia of the cross-section of the pinion} \)

\[ \xi = \frac{x}{f} \text{ where } x = \text{distance from free end of pinion} \]

\( F_s = \delta S = \text{load per unit axial strength} \).

If the average load per unit axial length is \( F_t \), the load concentration

\[ W_F = \frac{F_x}{F_t} = \delta \frac{S}{F_t} = \text{const.} \delta \]

![Fig. 2](image-url)
Without going into details the nature of distribution is shown in Fig. 24 for a very long pinion (I) and a shorter one (II). It can be seen that:

(a) the load concentration at the torque end is greater than at the free end;
(b) for the shorter pinion the concentration is less, also less pronounced;
(c) the load concentration at the middle is very low for the long pinion.

The nature of distribution depends on the values of \( a \) and \( \beta \) and the concentration is increasing with both quantities. It is obvious that for very long pinions like the ones used for turbine drives above analysis is of practical importance, but for shorter pinion the distribution tends to be uniform. However, an indication as to what limit shall be set to tooth length in order to obtain approximately uniform load distribution, can be easily found by suitably selecting \( a \) and \( \beta \) as follows:

\[
\beta = \frac{f^4 S}{E I}, \quad a = \frac{d^2 f^2 S}{4GJ},
\]

where \( f = \frac{r}{d} \) is the ratio of face width to pitch diameter.

\[
\sqrt{\frac{\beta E I}{a G J}} = \text{const.}
\]

From the discussion of above paper, it appears that the load distribution tends to be uniform if \( \beta \leq 9 \) and \( a \approx 2 \). Furthermore \( \frac{E}{G} = 2(1 + m) \) where \( m = \text{Poisson's ratio} = 0.25 \) to \( 0.3 \) for most gear materials.

To approximate \( I \) and \( J \) we take the moment and polar moment of inertia of a circle with diameter \( d \). Thus we have

\[
\beta = \frac{9}{2}, \quad \frac{E}{G} \approx 2.5, \quad \frac{I}{J} = \frac{1}{2}
\]

and

\[
\frac{f}{d} \leq \sqrt{\frac{4.5 \cdot 2.5}{8}} = 1.185
\]

Substituting \( d = \frac{f}{P} \) we obtain

\[
f = \lambda \cdot \frac{f}{P} \quad \text{where} \quad \lambda < 1.185.
\]

5. Acceleration and impact load in gear tooth action due to elasticity of gears, shafts, bearings and error in tooth spacing and involute profile. Limitations of British Standard gear design formulae.

Involute gears will transmit power continuously, perfectly smoothly without variation with uniform velocity only, if they have perfect tooth profiles, perfect spacing and are made of perfectly rigid material, mounted on perfectly rigid shafts and bearings. In such case, if it would exist, the sum total of tooth pressure between contacting teeth would be constant.

This, however, is an ideal condition which cannot be achieved. Due to elasticity of gears, shafts and bearings, the gear tooth will deform (compress and deflect). The amount of deformation is dependent on the point of application of the load which varies through a cycle. (Deflection due to bending increases as the point of application approaches the tip of the gear tooth.) Also, due to a contact ratio greater than unity, there will be periods within the cycle, when the load is shared between two pairs of teeth.

Hence, it follows that even if there is no error in tooth profile and spacing, due to this varying deformation, there will be variation of speed within a cycle, i.e. acceleration and deceleration causing acceleration forces due to the inertia of rotating masses.

Experiments show that the effect of this fluctuation will be mostly effective when the gear contact shifts from one pair to two pairs of gear teeth. In this
moment, however, a certain balance will be restored in tooth pressure because the second pair just entering contact will take a greater shear of the load on account of the increased deformation due to acceleration. Thus, the total tooth pressure will increase; however, the pressure on a single tooth probably will not exceed the static load.

Therefore with perfectly cut gears the dynamic effect could be neglected. 

However, there are always errors in both profile and spacing. This will in itself be responsible for a fluctuating speed in both the gear and the pinion, causing acceleration and deceleration forces. These forces will increase with increasing error and will depend also on the inertia of all rotating masses. The acceleration forces will depend also on the elasticity of gears, etc., and will tend to smooth out with increasing flexibility of the gear teeth.

If the acceleration is sufficiently great to cause separation of gear teeth, then in the deceleration period the teeth come in contact with impact causing high impact loads, i.e. high additional stresses. This impact which is again largely dependent on the error, may be several times the static load. Impact loads are caused usually by errors which exceed the deformation caused by the load.

It can be seen that due to the multitude of variables, the problem of calculating acceleration and impact loads in gearing, is a very complex one. A method developed by Earle Buckingham offers a possibility to calculate the: (1) effect of rotating masses; (2) acceleration load; (3) separation due to error and acceleration; and (4) impact load due to separation, if sufficient data is available.

It would be, however, very cumbersome to carry out these calculations in each case. If we consider that above effects are important only when—

(a) the speed is high;
(b) the errors in tooth action are appreciable,
we can easily decide in which cases is the great amount of labour, associated with above calculations, justified.

The British Standard gear design practice assumes an accuracy increasing with speed. Also the speed factor in addition to fatigue effect allows for the additional dynamic load due to limited speed variations. Practice has also justified the use of the British Standard Formulae which, due to their simplicity and good agreement with actual gear combination, can be safely used for all but extreme cases.

However, when speeds are very high in important drives the extra effort to check the additional loads and decide on a suitable accuracy is well justified.

It is also worth while to note that the gear noise is increasing with error and that very often the effect of error can be partially balanced by increased flexibility of pinion (use of plastic, etc., gears).

Noise in metallic gears is in most cases an indication that higher accuracy is needed while a comparatively silent drive indicates a satisfactory design as well as accuracy.

Fortunately, with continuously improving production methods, it is possible to-day to produce gears with very high accuracy without unduly increasing manufacturing costs.

In the following the B.S. horsepower equations will be given in a simplified form, taking in account the limitations with regard to face width as discussed previously. If we also keep in mind that finer pitch gears have more flexible teeth than coarse pitch gears, a proper balance can be found in the selection of the constant $e$ and the diametral pitch $P$.

We arrive at the simplified form of the horsepower equations as follows:—

Allowable horse power for strength

$$ hP_b = \frac{X_bS_b fY_{t2}n}{126000P^2} \text{ and with } f = \frac{c}{P}$$

$$ hP_b = \left( \frac{X_bS_b Y_{t2}n}{126000} \right) \frac{e}{P^2}$$

Since $X_b, S_b, Y, f, n$ are all known or selected at the outset, the first term in brackets is a constant, and can be calculated.
Thus we can write:

\[ C_{b_p} = \frac{X_{b_p} S_{b_p} Y_{b_p}}{126000} \text{ and } C_{b_w} = \frac{X_{b_w} S_{b_w} Y_{b_w}}{126000} \]

\[ kp_{b_p} = C_{b_p} c \left( \frac{P}{P^2} \right) \]

\[ kp_{b_w} = C_{b_w} c \left( \frac{P}{P^2} \right) \]

where \( c \leq 1 \)

Similarly for wear we have:

\[ kp_{c_p} = \frac{X_{c_p} S_{c_p} Y_{c_p}}{126000} \]

\[ kp_{c_w} = \frac{X_{c_w} S_{c_w} Y_{c_w}}{126000} \]

The four equations are:

Allowable horsepower for pinion strength = \( kp_{b_p} = C_{b_p} c \left( \frac{P}{P^2} \right) \) \( A \)

" " " " gear strength = \( kp_{b_w} = C_{b_w} c \left( \frac{P}{P^2} \right) \) \( A \)

" " " " pinion wear = \( kp_{c_p} = C_{c_p} c \left( \frac{P}{P^2} \right) \) \( B \)

" " " " gear wear = \( kp_{c_w} = C_{c_w} c \left( \frac{P}{P^2} \right) \) \( B \)

The design horsepower \( kp_d \) must be equated to the smallest of \( A \) from which \( \frac{c}{P^2} = \frac{kp_d}{C_b} \) and selecting a suitable \( c \) (say between 0.4 and 1.0). \( P \) can be calculated:

\[ P = \sqrt[3]{\frac{c C_b}{kp_d}} \]

Substituting the obtained value of \( P \) and the selected value of \( c \) in the smallest of \( B \), we must obtain \( kp_c \geq kp_d \). If, however, \( kp_c \) is less than \( kp_d \), the value of \( c \) or \( P \) must be adjusted. Alternatively, if \( kp_c \) is obviously smaller than \( kp_b \), \( P \) can be calculated from equations \( B \), and checked by \( A \).

**Example of Gear Calculation.**

Calculate the pitch (\( P \)) and face width (\( f \)) of a pair of spur gears which must be capable of transmitting 40 h.p. in a 12 hrs. per day service at constant speed.

The data given is as follows:

- Pinion speed = \( n = 500 \) r.p.m.
- Gear wheel speed = \( N = 100 \) r.p.m.
- Pressure angle = \( \psi = 20^\circ \)
- Material of pinion = 0.4% C. steel, normalised.
- Material of gear wheel = ordinary grade cast iron.

First we decide on the number of teeth:

If number of teeth in pinion = \( t = 20 \)

number of teeth in gear wheel = \( T = \frac{n \cdot t}{N} = 100 \)

Face width \( f = c \cdot \frac{t}{P} \)
To find the stress factors, strength factors, zone factor, and speed factors we use British Standard Specification: B.S. 436-1940, which will yield the following data:

1. Stress factors. (Pages 34-37.)
\[
\begin{align*}
S_{b_p} &= 5800 \\
S_{b_w} &= 1000 \\
S_{b_p}' &= 19000 \\
S_{b_w}' &= 1600
\end{align*}
\]

2. Zone factor. (Chart No. 8.)
\[
Z = 2.18
\]

3. Strength factors. (Chart No. 9.)
\[
Y_p = 0.72 \\
Y_w = 0.61
\]

4. Speed factors. (Combined running time and speed factors.)
From chart No. 10 (strength).
\[
X_{b_p} = 0.307 \\
X_{b_w} = 0.42
\]
From chart No. 11 (wear).
\[
X_{b_p}' = 0.307 \\
X_{b_w}' = 0.42
\]
The allowable horse power is the least of the following four quantities:

- Pinion strength: \( C_{b_p} \frac{P_{a}}{P_{a}} \)
- Gear strength: \( C_{b_w} \frac{C_{w}}{P_{a}} \)
- Pinion wear: \( C_{b_p}' \frac{P_{a}}{P_{a}} \)
- Gear wear: \( C_{b_w}' \frac{C_{w}}{P_{a}} \)

where
\[
\begin{align*}
C_{b_p} &= \frac{X_{b_p} S_{b_p} Y_{p} P_{a}}{126,000} = \frac{0.307 \times 19,000 \times 0.72 \times 20 \times 500}{126,000} = 6600 \\
C_{b_w} &= \frac{X_{b_w} S_{b_w} Y_{w} T.T.N}{126,000} = \frac{0.42 \times 5800 \times 0.61 \times 100 \times 20 \times 100}{126,000} = 2350 \\
C_{b_p}' &= \frac{X_{b_p}' S_{b_p}' Z.T.T.N}{126,000} = \frac{0.307 \times 1600 \times 2.18 \times 20 \times 500}{126,000} = 1700 \\
C_{b_w}' &= \frac{X_{b_w}' S_{b_w}' Z.T.T.N}{126,000} = \frac{0.42 \times 1000 \times 2.18 \times 100 \times 20 \times 100}{126,000} = 1450
\end{align*}
\]

Hence we use gear wheel wear to calculate the pitch.

Allowable hp gear wheel wear = 1450 \( \frac{c}{P_{a}} \) = 40

Using \( c = 0.75 \)
\[
p_{a} = \frac{0.75 \times 1450}{40} = 27.3
\]

2.8 \( \log P = \log 27.3 = 1.43616 \)

\( \log P = \frac{1.43616}{2.8} = 0.515 \)

\( P = 3.27 \)

Using the nearest standard pitch \( P = 3.25 \)

Checking for gear strength
\[
C_{b_w} \frac{c}{P_{a}} = 2350 \times 0.75 = 51 \frac{hp}{40} \text{ hence } P = 3.25 \text{ is satisfactory.}
\]

And the face width
\[
f = \frac{c \cdot f}{P} = \frac{0.75 \times 20}{3.25} = 4.625 = 4\frac{1}{4}"
(1) H. E. Merritt: Gears. (Sir Isaac Pitman & Sons Ltd., London, 1943.)
(2) E. Buckingham: Spur Gears. (McGraw-Hill, N.Y., 1928.)
(3) E. Buckingham: Manual of Gear Design. (Machinery, N.Y., 1935.)